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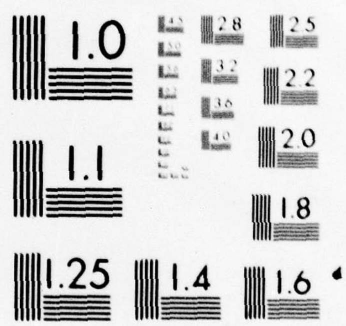

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A PENALTY PLATE-BENDING ELEMENT FOR THE ANALYSIS OF LAMINATED ANISOTROPIC COMPOSITE PLATES

See 1473 in back

by

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August 1979

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20. Abstract - Cont'd

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A PENALTY PLATE-BENDING ELEMENT FOR THE ANALYSIS OF LAMINATED ANISOTROPIC COMPOSITE PLATES

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SUMMARY

A C° (penalty) finite element is developed for the equations governing the heterogeneous laminated plate theory of Yang, Norris and Stavsky. The YNS theory is a generalization of Mindlin's theory for homogeneous, isotropic plates to arbitrarily laminated anisotropic plates and includes shear deformation and rotary inertia effects. The present element can also be used in the analysis of thin plates by appropriately specifying the penalty parameter. A variety of problems are solved, including those for which solutions are not available in the literature, to show the material effects and the parametric effects of plate aspect ratio, length-to-thickness ratio, lamination scheme, number of layers and lamination angle on the deflections, stresses, and vibration frequencies. Despite its simplicity, the present element gives very accurate results.

INTRODUCTION

Over the past few years composites, especially fiber-reinforced laminates, have found increasing application in many engineering structures. The fiber-reinforced composites possess two desirable features: one is their high stiffness-to-weight ratio, and the other is their anisotropic material property that can be tailored through variation of the fiber orientation

and stacking sequence--a feature which gives the designer an added degree of flexibility.

Recent developments in the analysis of plates laminated of fiber-reinforced materials indicate that the thickness effect on the behavior of the plate is more pronounced than in isotropic plates. The classical thin plate theory assumes that normals to the midsurface before deformation remain straight and normal to the midsurface after deformation, implying that transverse shear deformation effects are negligible. As a result, the free vibration frequencies, for example, calculated using the thin plate theory are higher than those obtained by the Mindlin plate theory[1], which includes transverse shear and rotary inertia effects; the deviation increases with increasing mode number. Higher order linear theories that include transverse shear effects have also appeared (see Reissner[2] and Lo, Christensen, and Wu[3]). Elasticity solutions by Pagano and his associates[4-7] indicate the inadequacy of the classical laminated plate theory (e.g., Reissner and Stavsky[8], Dong et al.[9], and Bert and Mayberry[10], in which the classical Kirchhoff-Love kinematic assumptions are adopted and effects of transverse shear deformations are neglected. The transverse shear deformation effects are even more pronounced, due to the low transverse shear modulus relative to the in-plane Young's moduli, in the case of filamentary composite plates. A reliable prediction of the response characteristics of high modulus composite plates requires the use of shear deformable theories.

A number of shear deformable theories for laminates have been proposed to date. The first such theory for laminated isotropic plates is due to Stavsky[11]. The theory has been generalized to laminated anisotropic plates

by Yang, Norris, and Stavsky[12]. A review of various other theories, for example, the effective stiffness theory of Sun and Whitney[12], the higher-order theory of Whitney and Sun[14], and the three-dimensional elasticity theory of Srinivas and Rao[15], can be found in the paper by Bert[16]. Other approximate theories that have been proposed in the literature include the refined laminated plate theory of Mau[17], the continuum theory of Sun, Achenbach, and Herrmann[18], and diffusing continuum theory of Bedford and Stern[19] which were primarily developed for use in wave-propagation problems. It has been shown, see for example, the papers by Sun and Whitney[13] and Srinivas and Rao[15], that the Yang-Norris-Stavsky (YNS) theory is adequate for predicting the overall behavior such as transverse deflections and natural frequencies (first few modes) of laminated anisotropic plates.

The first application of the YNS theory is apparently due to Whitney and Pagano[20], who considered cylindrical bending of antisymmetric cross-ply and angle-ply plate strips under sinusoidal load distribution and free vibration of antisymmetric angle-ply plate strips. Fortier and Rosettos[21] analyzed free vibration of thick rectangular plates of unsymmetric cross-ply construction while Sinha and Rath[22] considered both vibration and buckling for the same type of plates. Recently, Bert and Chen[23] presented, using the YNS theory, a closed-form solution for the free vibration of simply supported rectangular plates of antisymmetric angle-ply laminates.

While considerable effort has been expended in the finite-element analysis of isotropic plates, only limited investigations of laminated anisotropic plates can be found in the literature. Pryor and Barker[24], and Barker, Lin and Dara[25] used the conventional displacement finite-element

method to analyze thick laminated plates. The element has seven degrees of freedom (three displacements, two rotations, and two shear rotations) per node. Exploiting the symmetries exhibited by anisotropic plates, Noor and Mathers[26-28] studied the effects of shear deformation and anisotropy on the accuracy and convergence of several shear-flexible displacement finite element models based on a form of Reissner's plate theory. The analysis was limited to symmetrically laminated cross-ply plates and the element used involved 80 degrees of freedom per element. The conventional finite element, when applied to relatively thick laminated plates, either has failed to predict accurately the local deformations and stresses of a plate under bending or is too expensive to use due to the large number of degrees of freedom involved for even relatively simple problems. Mau and Witmer [29]and Mau, Tong, Pian[30] have employed the so-called hybrid-stress finite-element method to analyze composite plates including shear deformation. The hybrid elements have proven (see Gallagher[31]) to have some convergence problems, and in some cases they give erroneous results. Most recently, Panda and Natarajan[32] used, following Mawenya and Davies[33], the quadratic shell element of Ahmad, Irons and Zienkiewicz[34] with the same normal rotation through the thickness to claim improved accuracy over Mawenya and Davies[33]. The 'thickness concept' mentioned there is essentially the same as that used in the YNS theory[12]. The authors were primarily concerned with the accuracy of the element, and no attempt was made to solve new problems for which there do not exist any closed-form or exact solutions.

The present paper is concerned with the development of a simple C° element for YNS theory of laminated composite plates. The penalty function

concept of Courant[35] (also see Zienkiewicz[36]) is used to develop the finite element model. The element contains five degrees of freedom, three displacements and two slopes (i.e. shear rotations), per node. The accuracy of the element is demonstrated via problems for which the exact solutions and numerical results are available, and results are also presented for a variety of problems for which solutions are not available in the literature.

LAMINATED PLATE THEORY OF YANG-NORRIS-STAVSKY (YNS)

Consider a plate of constant thickness h composed of a finite number, L, of thin anisotropic layers oriented at angles $\theta_1, \theta_2, \ldots, \theta_L$. The origin of the coordinate system is located within the middle plane (x-y) with the z-axis being normal to the mid-plane. The material of each layer is assumed to possess a plane of elastic symmetry parallel to the xy-plane. We shall denote the middle plane with Ω.

The YNS theory is based on the following assumed displacement field:

$$\begin{aligned} u &= u_o(x,y,t) + z\psi_x(x,y,t) \\ v &= v_o(x,y,t) + z\psi_y(x,y,t) \qquad (1) \\ w &= w(x,y,t) \end{aligned}$$

where u, v, and w are the displacement components in the x, y, and z directions, respectively, t is the time, u_o and v_o are the in-plane (stretching) displacements of the middle plane, and ψ_x and ψ_y are the shear rotations. Recalling the strain-displacement equations of linear elasticity, we have

$$\epsilon_x = \frac{\partial u_o}{\partial x} + z\,\frac{\partial \psi_x}{\partial x}\ , \quad \epsilon_y = \frac{\partial v_o}{\partial y} + z\,\frac{\partial \psi_y}{\partial y}\ , \quad \epsilon_z = 0$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)\ , \tag{2}$$

$$\gamma_{xz} = \psi_x + \frac{\partial w}{\partial x}\ , \quad \gamma_{yz} = \psi_y + \frac{\partial w}{\partial y}$$

Owing to the existence of a plane of elastic symmetry, the constitutive relations for any layer in the (x,y) system are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{yz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{22} & 0 & 0 & Q_{26} \\ 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & Q_{45} & Q_{55} & 0 \\ Q_{16} & Q_{26} & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} . \tag{3}$$

where Q_{ij} are the (material) stiffness components.

Introducing the stress and moment resultants per unit length,

$$(N_1,N_2,N_6) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})\, dz\ , \quad (Q_x, Q_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz})\, dz$$

$$(M_1,M_2,M_6) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})\, zdz \tag{4}$$

we can write (2) and (3) in terms of the resultants and displacements:

$$\begin{bmatrix} N_1 \\ N_2 \\ Q_y \\ Q_x \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & 0 & A_{16} & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 & 0 \\ A_{16} & A_{26} & 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & 0 & 0 & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & 0 & 0 & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & 0 & 0 & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} u_{o,x} \\ v_{o,y} \\ w_{,y} + \psi_y \\ w_{,x} + \psi_x \\ u_{o,y} + v_{o,x} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{bmatrix} \quad (5)$$

The material componets A_{ij}, B_{ij}, and D_{ij} are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{(m)} (1,z,z^2)\, dz \ , \quad (i,j = 1,2,6)$$

$$A_{ij} \equiv k_\alpha k_\beta \bar{A}_{ij} \ , \quad \bar{A}_{ij} = \int_{-h/2}^{h/2} Q_{ij}^{(m)} dz \ , \quad (i,j = 4,5) \ , \ \alpha=6-i, \ \beta=6-j \quad (6)$$

The stiffness coefficients $Q_{ij}^{(m)}$ depend on the material properties and orientation of the m-th layer. The parameters k_i are the shear correction coefficients.

The equations of motion associated with YNS theory are

$$\frac{\partial N_1}{\partial x} + \partial \frac{\partial N_6}{\partial y} = p \frac{\partial^2 u}{\partial t^2} + R \frac{\partial^2 \psi_x}{\partial t^2}$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = p \frac{\partial^2 v}{\partial t^2} + R \frac{\partial^2 \psi_y}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = p \frac{\partial^2 w}{\partial t^2} - P$$

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_x = I \frac{\partial^2 \psi_x}{\partial t^2} + R \frac{\partial^2 u}{\partial t^2} \tag{7}$$

$$\frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_y = I \frac{\partial^2 \psi_y}{\partial t^2} + R \frac{\partial^2 v}{\partial t^2}$$

where

$$(p,R,I) = \int_{-h/2}^{h/2} (1,z,z^2)\, \rho^{(m)} dz \tag{8}$$

$\rho^{(m)}$ being the material density of layer m, and $P = P(x,y)$ is the transversely distributed load.

The strain energy and the kinetic energy, for a fixed time t, are given by

$$U = U_1 + U_2 \tag{9}$$

$$T = \frac{1}{2}\int_{\Omega} \{p[(\frac{\partial u}{\partial t})^2 + (\frac{\partial v}{\partial t})^2 + (\frac{\partial w}{\partial t})^2] + I[(\frac{\partial \psi_x}{\partial t})^2 + (\frac{\partial \psi_y}{\partial t})^2]$$
$$+ 2R[\frac{\partial \psi_x}{\partial t}\frac{\partial u}{\partial t} + \frac{\partial \psi_y}{\partial t}\frac{\partial v}{\partial t}]\}\, dxdy \tag{10}$$

where

$$U_1 = \frac{1}{2}\int_{\Omega} \{A_{11} (\frac{\partial u_o}{\partial x})^2 + 2A_{16} \frac{\partial u_o}{\partial x}\frac{\partial u_o}{\partial y} + A_{66} (\frac{\partial u_o}{\partial y})^2 + (A_{12} \frac{\partial v_o}{\partial y} + A_{16}\frac{\partial v_o}{\partial x}) \frac{\partial v_o}{\partial y}$$
$$+ \frac{\partial v_o}{\partial y} (A_{12} \frac{\partial u_o}{\partial x} + A_{26} \frac{\partial u_o}{\partial y}) + \frac{\partial u_o}{\partial y} (A_{26} \frac{\partial v_o}{\partial y} + A_{66} \frac{\partial v_o}{\partial x})$$
$$+ \frac{\partial v_o}{\partial x} (A_{16}\frac{\partial u_o}{\partial x} + A_{66} \frac{\partial u_o}{\partial y}) + A_{22} (\frac{\partial v_o}{\partial y})^2 + 2A_{16} \frac{\partial v_o}{\partial y}\frac{\partial v_o}{\partial x}$$
$$+ A_{66} (\frac{\partial v_o}{\partial x})^2 + \frac{\partial u_o}{\partial x} (B_{11} \frac{\partial \psi_x}{\partial x} + B_{16} \frac{\partial \psi_x}{\partial y} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \frac{\partial \psi_y}{\partial x})$$
$$+ (\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x})(B_{16} \frac{\partial \psi_x}{\partial x} + B_{66} \frac{\partial \psi_x}{\partial y} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{66} \frac{\partial \psi_y}{\partial x})$$

$$+ \frac{\partial v_o}{\partial y} (B_{12} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_x}{\partial y} + B_{22} \frac{\partial \psi_y}{\partial y} + B_{26} \frac{\partial \psi_y}{\partial x}) + \frac{\partial \psi_x}{\partial x} (B_{11} \frac{\partial u_o}{\partial x}$$

$$+ B_{16} \frac{\partial u_o}{\partial y} + B_{12} \frac{\partial v_o}{\partial y} + B_{16} \frac{\partial v_o}{\partial x}) + (\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x})(B_{16} \frac{\partial u_o}{\partial x} + B_{66} \frac{\partial u_o}{\partial y}$$

$$+ B_{26} \frac{\partial v_o}{\partial y} + B_{66} \frac{\partial v_o}{\partial x}) + \frac{\partial \psi_y}{\partial y} (B_{12} \frac{\partial u_o}{\partial x} + B_{26} \frac{\partial u_o}{\partial y} + B_{22} \frac{\partial v_o}{\partial y} + B_{26} \frac{\partial v_o}{\partial x})$$

$$+ D_{11} (\frac{\partial \psi_x}{\partial x})^2 + 2D_{16} \frac{\partial \psi_x}{\partial x} \frac{\partial \psi_x}{\partial y} + D_{66} (\frac{\partial \psi_x}{\partial y})^2 + \frac{\partial \psi_x}{\partial x} (D_{12} \frac{\partial \psi_y}{\partial y} + 2D_{16} \frac{\partial \psi_y}{\partial x})$$

$$+ \frac{\partial \psi_y}{\partial y} (D_{12} \frac{\partial \psi_x}{\partial x} + 2D_{26} \frac{\partial \psi_x}{\partial y}) + 2D_{66} \frac{\partial \psi_x}{\partial y} \frac{\partial \psi_y}{\partial x} + D_{22} (\frac{\partial \psi_y}{\partial y})^2$$

$$+ 2D_{26} \frac{\partial \psi_y}{\partial x} \frac{\partial \psi_y}{\partial y} + D_{66} (\frac{\partial \psi_y}{\partial x})^2 \} \, dxdy \tag{11}$$

$$U_2 = \frac{1}{2} \int_\Omega \{[A_{44} (\frac{\partial w}{\partial x} + \psi_x) + A_{45} (\frac{\partial w}{\partial y} + \psi_y)] (\frac{\partial w}{\partial x} + \psi_x) +$$

$$[A_{45} (\frac{\partial w}{\partial x} + \psi_x) + A_{55} (\frac{\partial w}{\partial y} + \psi_y)] (\frac{\partial w}{\partial y} + \psi_y)\} \, dxdy \tag{12}$$

Note that the quantities in the square brackets of U_2 are the shear forces Q_x and Q_y, respectively.

PENALTY FUNCTION FORMULATION OF THE EQUATIONS

The assumption of the classical thin-plate theory that the normals to the midsurface before deformation remain straight and normal to the midsurface after deformation implies that

$$\psi_x = - \frac{\partial w}{\partial x} \quad , \text{ and } \quad \psi_y = - \frac{\partial w}{\partial y} \tag{13}$$

If we substitute for ψ_x and ψ_y from (13) into (11), we obtain the strain energy $U = U_1$ associated with the classical thin-plate theory. In that case U_1 involves the second-order derivatives of the transverse deflection, and the associated (conventional) finite-element formulation results in

complicated elements (with many degrees of freedom). Approaches that have been taken to relax the continuity requirements placed on the shape functions in the displacement formulation of the thin-plate theory include, in addition to the nonconforming, hybrid and mixed formulations, the "discrete Kirchhoff hypothesis" of Wempner, Oden and Kross[37], Fried[38], and the "residual energy balancing" and "reduced integration" techniques of Fried[39], Zienkiewicz, Taylor and Too[40], and Hughes, Taylor and Kanoknukulchai[41]. The present penalty function method is a formalization and extension of these ideas to the shear deformable theory of laminated composite plates.

The problem of finding the static solution (u,v,w) to the thin plate equations can be viewed as one of finding the critical points of the total potential energy $\pi_1 = U_1 + V$, where U_1 is the strain energy given by (11) in terms of u, v and w, and V is the potential energy due to applied loads. Alternately, the problem can also be viewed as one of finding (u,v,ψ_x,ψ_y) subject to the constraint conditions in (13). To incorporate the constraints, one can use the Lagrange multiplier method, or the penalty function method.

If the Lagrange multiplier method is used, we have

$$U_L = U_1 + \int_\Omega [\lambda_x (\frac{\partial w}{\partial x} + \psi_x) + \lambda_y (\frac{\partial w}{\partial y} + \psi_y)]\, dxdy \tag{14}$$

where λ_x and λ_y are the Lagrange multipliers. Comparing the Euler equations of $\pi_L = U_L + V$ with those of $\pi = U + V$, we see that the Lagrange multipliers are given by,

$$\begin{aligned} \lambda_x &= Q_x \equiv A_{44} (\frac{\partial w}{\partial x} + \psi_x) + A_{45} (\frac{\partial w}{\partial y} + \psi_y) \\ \lambda_y &= Q_y \equiv A_{45} (\frac{\partial w}{\partial x} + \psi_x) + A_{55} (\frac{\partial w}{\partial y} + \psi_y) \end{aligned} \tag{15}$$

Thus, U_L is equivalent to the strain energy U of the shear deformable theory.

If the penalty function method is used, the modified functional is given by $\pi_p = U_1 + U_p + V$, wherein the penalty functional U_p is chosen to be

$$U_p = \frac{1}{2}\int_\Omega [\epsilon_1^2 (\frac{\partial w}{\partial x} + \psi_x)^2 + \epsilon_2^2 (\frac{\partial w}{\partial y} + \psi_y)^2 + 2\epsilon_1\epsilon_2 (\frac{\partial w}{\partial x} + \psi_x)(\frac{\partial w}{\partial y} + \psi_y)]\, dxdy \quad (16)$$

where ϵ_1^2 and ϵ_2^2 are the penalty parameters. Clearly, in the limits $\epsilon_1, \epsilon_2 \to \infty$, the constraints are satisfied exactly. As opposed to the Lagrange multiplier method the constraints are satisfied only approximately, and no additional variables are introduced in the penalty method. Comparing the Euler equations of the functional π_p with the equations of the YNS theory, we see the correspondence,

$$Q_x \approx Q_x^\epsilon \equiv \epsilon_1^2(\frac{\partial w}{\partial x} + \psi_x) + \epsilon_1\epsilon_2 (\frac{\partial w}{\partial y} + \psi_y)$$

$$Q_y \approx Q_y^\epsilon \equiv \epsilon_2\epsilon_1 (\frac{\partial w}{\partial x} + \psi_x) + \epsilon_2^2(\frac{\partial w}{\partial y} + \psi_y) \quad (17)$$

$$\epsilon_1^2 \approx k_1^2 \bar{A}_{44}\ , \quad \epsilon_1\epsilon_2 \approx k_1 k_2 \bar{A}_{45}\ , \quad \epsilon_2^2 \approx k_2^2 \bar{A}_{55}$$

This correspondence implies that for very large values of ϵ_i, the equations govern the thin-plate theory, and for values of ϵ_i given in (17), the equations coincide with the YNS theory.

FINITE-ELEMENT MODELS

Here we present a (semidiscrete) finite-element model based on $\pi_p(u,v,\psi_x,\psi_y,w)$. We assume, over each element Ω_e, the same kind of interpolation for all of the variables,

$$u_o^e = \sum_i^n u_i^e N_i^e\ , \quad v_o^e = \sum_i^n v_i^e N_i^e\ , \quad \text{etc.} \quad (n = \text{nodes per element}) \quad (18)$$

where N_i^e are the element interpolation (or shape) functions, and u_i^e, and v_i^e are the nodal values of u_o^e and v_o^e, respectively. Substituting (18) into the first variation of $\pi_p^e\ (u,v,w,\psi_x,\psi_y)$, and collecting the coefficients of the variations, δu_i, δv_i, etc., we obtain

$$[M^e]\{\ddot{\Delta}^e\} + [K^e]\{\Delta^e\} = \{F^e\} \tag{19}$$

where

$$\{\Delta^e\} = \begin{Bmatrix} \{u^e\} \\ \{v^e\} \\ \{w^e\} \\ \{\psi_x^e\} \\ \{\psi_y^e\} \end{Bmatrix}, \quad [M] = \begin{bmatrix} p[S^o] & & & R[S^o] & \\ & p[S^o] & & & R[S^o] \\ & & p[S^o] & & \\ & \text{symmetric} & & I[S^o] & \\ & & & & I[S^o] \end{bmatrix} \tag{20}$$

The elements $K_{ij}^{\alpha\beta}$ $(\alpha,\beta=1,2,\ldots,5;\ i,j=1,2,\ldots,n)$ of the stiffness matrix, S_{ij}^o of the mass matrix are given by

$$K_{ij}^{11} = A_{11}\ S_{ij}^{x} + A_{16}\ (S_{ij}^{xy} + S_{ji}^{xy}) + A_{66}\ S_{ij}^{y}$$

$$K_{ij}^{12} = A_{12}\ S_{ij}^{xy} + A_{16}\ S_{ij}^{x} + A_{26}\ S_{ij}^{y} + A_{66}\ S_{ji}^{xy}$$

$$K_{ij}^{14} = B_{11}\ S_{ij}^{x} + B_{16}\ (S_{ij}^{xy} + S_{ji}^{xy}) + B_{66}\ S_{ij}^{y}$$

$$K_{ij}^{15} = B_{12}\ S^{xy} + B_{16}\ S_{ij}^{x} + B_{26}\ S_{ij}^{y} + B_{66}\ S_{ji}^{xy}$$

$$K_{ij}^{22} = A_{26}\ (S_{ij}^{xy} + S_{ji}^{xy}) + A_{22}\ S_{ij}^{y} + A_{66}\ S_{ij}^{x}$$

$$K_{ij}^{24} = B_{16}\ S_{ij}^{x} + B_{66}\ S_{ij}^{xy} + B_{12}\ S_{ji}^{xy} + B_{26}\ S_{ij}^{y}$$

$$K_{ij}^{25} = B_{26}\ (S_{ij}^{xy} + S_{ij}^{xy}) + B_{66}\ S_{ij}^{x} + B_{22}\ S_{ij}^{y}$$

$$K_{ij}^{33} = \epsilon_1^2\, S_{ij}^{x} + \epsilon_2^2\, S_{ij}^{y} + \epsilon_1\epsilon_2\, (S_{ij}^{xy} + S_{ij}^{xy})$$

$$K_{ij}^{34} = \epsilon_1^2\, S_{ij}^{xo} + \epsilon_1\epsilon_2\, S_{ij}^{yo} \;,\quad K_{ij}^{35} = \epsilon_1\epsilon_2\, S_{ij}^{xo} + \epsilon_2^2\, S_{ij}^{yo}$$

$$K_{ij}^{44} = D_{11}S_{ij}^{x} + D_{16}(S_{ij}^{xy} + S_{ji}^{xy}) + D_{66}S_{ij}^{y} + \epsilon_1^2\, S_{ij}^{o}$$

$$K_{ij}^{45} = D_{12}S_{ij}^{xy} + D_{66}S_{ji}^{xy} + D_{16}S_{ij}^{x} + D_{26}S_{ij}^{y} + \epsilon_1\epsilon_2\, S_{ij}^{o}$$

$$K_{ij}^{55} = D_{26}(S_{ij}^{xy} + S_{ji}^{xy}) + D_{66}S_{ij}^{x} + D_{22}S_{ij}^{y} + \epsilon_2^2\, S_{ij}^{o}$$

$$K_{ij}^{13} = K_{ij}^{23} = 0 \;,\quad S_{ij}^{\xi\eta} = \int_{\Omega^e} N_{i,\xi}N_{j,\eta}\, dxdy \;,\quad (\xi,\eta{=}0,x,y)$$

$$F_i^3 = \int_{\Omega^e} P\, N_i\, dxdy \;,\quad F_i^1 = F_i^2 = F_i^4 = F_i^5 = 0 \tag{21}$$

For free vibration, equation (19) becomes

$$([K^e] - \omega^2[M^e])\, \{\Delta^e\} = \{0\} \tag{22}$$

where ω is the frequency of the natural vibration. For static analysis, $\{\ddot{\Delta}\}$ is set to zero. The element stiffness matrices are assembled in the usual manner, and boundary conditions of the problem are imposed before solving for $\{\Delta\}$ or ω_n.

In the present study linear (n=4) and quadratic (n=8) elements of the serendepity family are used. The element stiffness matrices for these elements are of order 20x20, and 40x40, respectively.

NUMERICAL EXAMPLES AND DISCUSSION

The (penalty) finite element developed herein was employed in the bending and free vibration analyses of a variety of layered composite rectangular plates. All of the numerical results presented here were obtained using a uniform mesh of 2x2 quadratic (i.e. 8-node quadrilateral) elements in the quarter plate. Computations were conducted on an IBM 370/158 computer in single precision.

In the following analyses two types of boundary conditions, simply supported and clamped, and two types of orthotropic materials were used. The coordinate system and boundary conditions are shown in Figure 1. The material properties used are ($G_{12} = G_{13}$; $\nu_{12} = \nu_{13}$),

Material I: $E_1/E_2 = 25$, $G_{12}/E_2 = 0.5$, $G_{23}/E_2 = 0.2$, $\nu_{12} = 0.25$.

Material II: $E_1/E_2 = 40$, $G_{12}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = 0.25$

A value of 5/6 was used for the shear correction coefficients (see Whitney[42]).

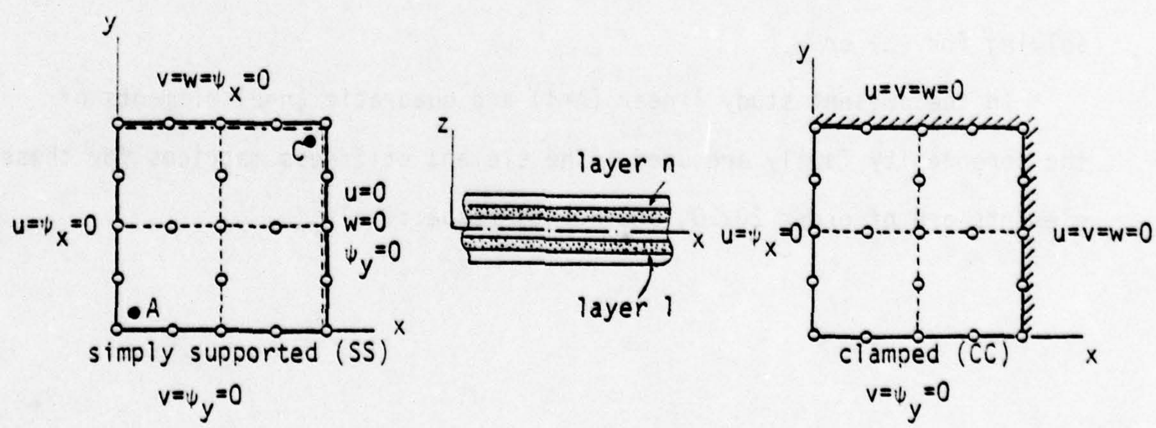


Figure 1 Coordinate system, finite-element mesh, and boundary conditions

Bending Analysis

First the effect of the reduced integration on the bending deflection and stresses is examined using a four-layer, cross-ply (0°/90°/90°/0°), simply supported square plate (Material II) subjected to sinusoidal loading (SSL),

$$P = P_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$$

The percentage error (between the solution obtained by using 2x2 Gauss points and 3x3 Gauss points) in the center deflection and maximum normal stress ($\bar{\sigma}_x = \bar{\sigma}_y$) as a function of the side-to-thickness ratio (a/h) are shown in Figure 2. The stresses were computed at the Gaussian points using equation (3). Figure 3 shows the bending deflection versus the side-to-thickness ratio for the same problem using 2x2 Gauss rule. This result is in excellent agreement with the closed-form solution of Whitney[43]. Thus, the standard 3x3 Gauss rule (for the numerical integration of elements in equation (21)) gives less accurate results, especially for ratios a/h > 10. Guided by this observation (also, see Zienkiewicz et al.[40]) the remaining results were obtained using 2x2 Gauss rule.

Figure 3 also shows the stresses, $\bar{\sigma}_x$, $\bar{\sigma}_y$, and $\bar{\tau}_{xy}$ for the four-layer, cross-ply (0°/90°/90°/0°), simply supported square plate under sinusoidal loading. To see the effect of loading and material on the deflection, the same problem was solved using Material II and uniform loading, and Material I and sinusoidal loading. Note that decreasing the ratio E_1/E_2 from 40 to 25 has the same effect as using the uniform loading in place of sinusoidal loading. Bending deflections and stresses are presented in Table 1 for a (4-ply (0°/90°/90°/0°), Material I) clamped plate under sinusoidal loading, and simply supported plate under point load at the center.

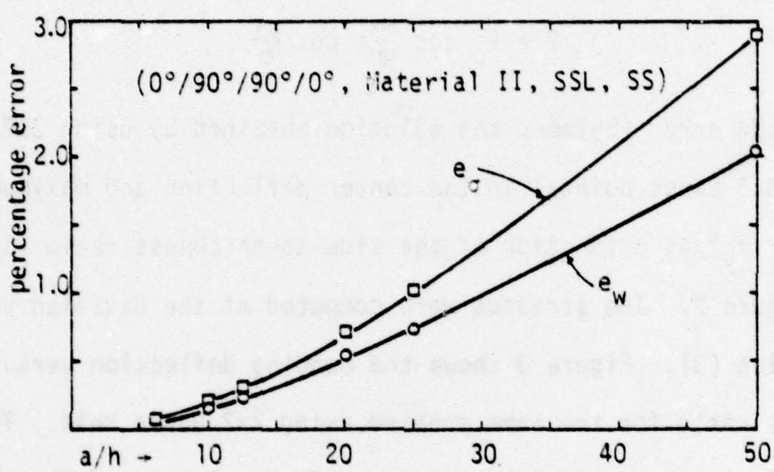


Figure 2 Effect of reduced integration on the bending deflections and stresses

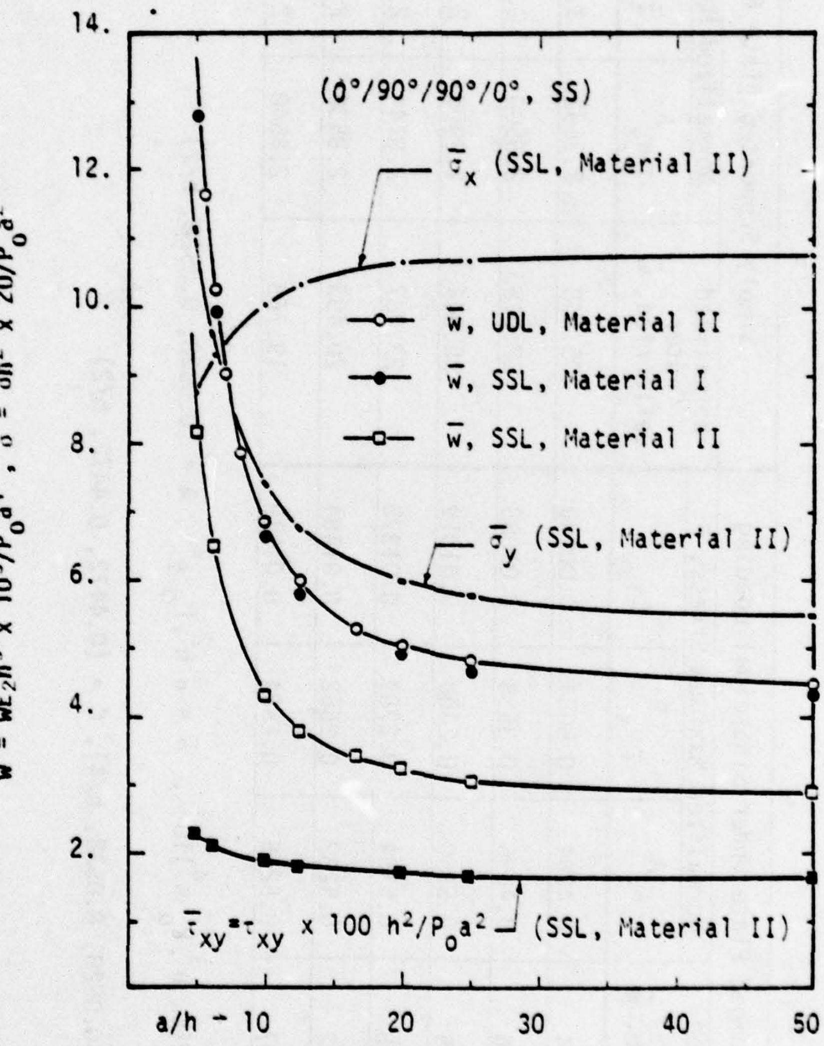


Figure 3 Bending deflections, and stresses vs. side to thickness ratio for four-layer, cross-ply, simply supported square plates

Table 1 Four-Layer (0°/90°/90°/0°) Square Plate (Material I)

a/h	Clamped Plate Under Sinusoidal Loading				Simply Supported Plate Under Point Load			
	Normalized Center Deflection, $\bar{w}$	Normalized Maximum Stress, $\bar{\sigma}$			Normalized Center Deflection, $\bar{w}$	Normalized Maximum Stress, $\bar{\sigma}$		
		$\mp\sigma_x$ A	$\mp\sigma_x$ B	$\pm\tau_{xy}$ C		$\mp\sigma_x$ A	$\mp\sigma_x$ B	$\pm\tau_{xy}$ C
5	12.9544	0.4284	0.5084	0.00900	105.448	2.3692	3.4564	0.0874
10	6.6956	0.4896	0.3538	0.01083	42.015	2.6500	2.7160	0.0765
20	4.9415	0.5150	0.2880	0.01319	25.344	2.7900	2.3258	0.0712
25	4.7208	0.5184	0.2787	0.01378	23.287	2.8144	2.2560	0.07029
50	4.4222	0.5232	0.2662	0.01484	20.493	2.8536	2.1296	0.06904
100	4.3567	0.5255	0.2634	0.01519	19.763	2.8640	2.0590	0.06916

$\bar{w} = W(E_2\, h^3/P_o\, a^4)10^3$, $\bar{\sigma} = \sigma\, h^2/P_o\, a^2$, A = (0.0528, 0.0528, h/2)

B = (0.0528, 0.0528, h/4), C = (0.4472, 0.4472, h/2)

To further illustrate the accuracy of the present element, two problems for which exact[5,7] and finite element solutions[32,33] are available, were solved and results are summarized in Tables 2 and 3 (for Material I). Table 2 contains the normalized bending deflections and stresses for three-layer, cross-ply (0°/90°/0°), simply supported square plate subjected to sinusoidal loading. The outer layers are each h/4, and the middle layer is h/2 thick (i.e. sandwich construction). Table 3 contains similar information for three-layer (equal thickness), cross-ply (0°/90°/0°), simply supported rectangular (b/a = 3) plate under sinusoidal loading. Present solutions are compared with exact solutions of Pagano[5] and Pagano and Hatfield[7], and the finite-element solutions of Panda and Natarajan[32] and Maweny and Davies[33]. It is clear that the present solution is the closest to the exact solution for the deflection for all ratios of a/h. Since the stresses in the present study are computed at the Gaussian points, it is not meaningful to compare for relative accuracy.

Figure 4 shows the normalized bending deflections versus the ratio a/h for the 3-layer, cross-ply, simply supported square plate under sinusoidal loading. For Material I the same problem was resolved with layer orientation of 0°/91°/0° (the middle layer is now oriented at 91°) to see the effect of slight variation (introduced, say, in manufacturing) in the orientation of the layers on the deflection. Note that the error in the angle causes slight variation in the deflection only at higher values of a/h (i.e. for thin plates).

Figure 5 shows plots of bending deflection versus the side-to-thickness ratio for twy-layer, cross-ply (0°/90°) square plate (Material II) under sinusoidal and uniform loadings, and for four-layer, symmetric angle-ply

Table 2 Three-layer (0°/90°/0°) simply supported square plate subjected to *sinusoidal loading* (Material I, $t_1 = t_3 = h/4$, $t_2 = h/2$)

a/h	Source	Normalized center deflection $\overline{w}$	Normal stresses, $\overline{\sigma}$ (top and bottom)*		
			$\mp\sigma_x(0,0,h/2)$	$\mp\sigma_y(0,0,h/4)$	$\pm\tau_{xy}(\frac{a}{2},\frac{b}{2},\frac{h}{2})$
5.0	Present FEM	2.9642	0.4196	0.5000	0.02804
6.25	Present FEM	2.2998	0.4442	0.4431	0.02629
10	Pagano & Hatfield[7]	1.709	0.559	0.403	0.0276
	Present FEM	1.5340	0.4842	0.3509	0.02342
	Panda & Natarajan[32]	1.448	0.532	0.307	0.0250
	Mawenya & Davies[33]	2.034	0.542	--	0.0292
12.5	Present FEM	1.3465	0.4965	0.3223	0.02241
20	Pagano & Hatfield[7]	1.189	0.543	0.309	0.0230
	Present FEM	1.1364	0.5118	0.2870	0.02144
	Panda & Natarajan[32]	1.114	0.557	0.307	0.0231
	Mawenya & Davies[33]	1.273	0.546	--	0.0239
25	Present FEM	1.0866	0.5154	0.2779	0.02115
50	Pagano & Hatfield[7]	1.031	0.539	0.276	0.0216
	Present FEM	1.0197	0.5208	0.2656	0.02077
	Panda & Natarajan[32]	1.016	0.565	0.287	0.0225
	Mawenya & Davies[33]	1.048	0.550	--	0.0221
100	Pagano & Hatfield[7]	1.008	0.539	0.271	0.0214
	Present FEM	1.0055	0.5235	0.2630	0.02073
	Panda & Natarajan[32]	1.003	0.566	0.284	0.0223
	Mawenya & Davies[33]	1.015	0.551	--	0.0213
Classical plate theory		1.000	0.539	0.269	0.0213

$\overline{w} = w\alpha(h^3/P_0a^4)$, $\overline{\sigma} = \sigma h^2/P_0a^2$, $\alpha = \{4G_{12} + [E_1 + (1+\nu_{12})E_2]/(1-\nu_{12}\nu_{21})\}\ \pi^4/12$

* Computed at the Gaussian points in the present study .

Table 3 Three-layer (0°/90°/0°) simply supported rectangular plate (b/a = 3) subjected to sinusoidal loading (Material I)

a/h	Source	Normalized center deflection, $\bar{w}$	Normalized stress, $\bar{\sigma}$ (top and bottom)*		
			$\mp\sigma_x(0,0,h/2)$	$\mp\sigma_y(0,0,h/6)$	$\pm\tau_{xy}(\frac{a}{2},\frac{b}{2},\frac{h}{2})$
5	Present FEM	1.695	0.5984	0.0691	0.01789
6.25	Present FEM	1.267	0.6006	0.0540	0.01338
10	Exact: Pagano[5]	0.919	0.725	0.0435	0.0123
	Present FEM	0.802	0.6031	0.0364	0.01017
	Panda & Natarajan[32]	0.752	0.653	0.0367	0.0105
	Mawenya & Davies[33]	1.141	0.685	--	0.0141
12.5	Present FEM	0.694	0.6038	0.0322	0.00941
20	Exact: Pagano[5]	0.610	0.650	0.0299	0.0093
	Present FEM	0.578	0.6045	0.0276	0.00858
	Panda & Natarajan[32]	0.565	0.654	0.0287	0.0091
	Mawenya & Davies[33]	0.664	0.651	--	0.0099
25	Present FEM	0.551	0.6046	0.0264	0.00838
50	Exact: Pagano[5]	0.520	0.628	0.0259	0.0084
	Present FEM	0.515	0.6044	0.0251	0.00812
	Panda & Natarajan[32]	0.513	0.654	0.0264	0.0087
	Mawenya & Davies[33]	0.529	0.640	--	0.0087
100	Exact: Pagano[5]	0.508	0.624	0.0253	0.0083
	Present FEM	0.506	0.6034	0.0253	0.00802
	Panda & Natarajan[32]	0.505	0.654	0.0261	0.0086
	Mawenya & Davies[33]	0.510	0.638	--	0.0085
Classical plate theory		0.503	0.623	0.0252	0.0083

$\bar{w} = w\ 100\ E_2h^3/P_oa^4$, $\bar{\sigma} = \sigma h^2/P_oa^2$

* Computed at the Gaussian points in the present study.

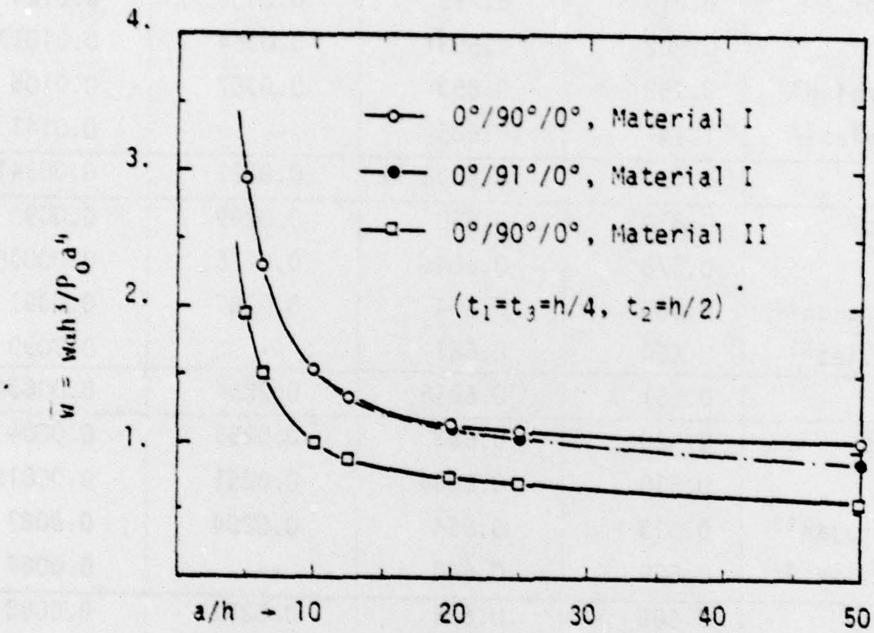


Figure 4 Bending deflection vs. side-to-thickness ratio for 3-layer, cross-ply, simply supported square plate under sinusoidal loading

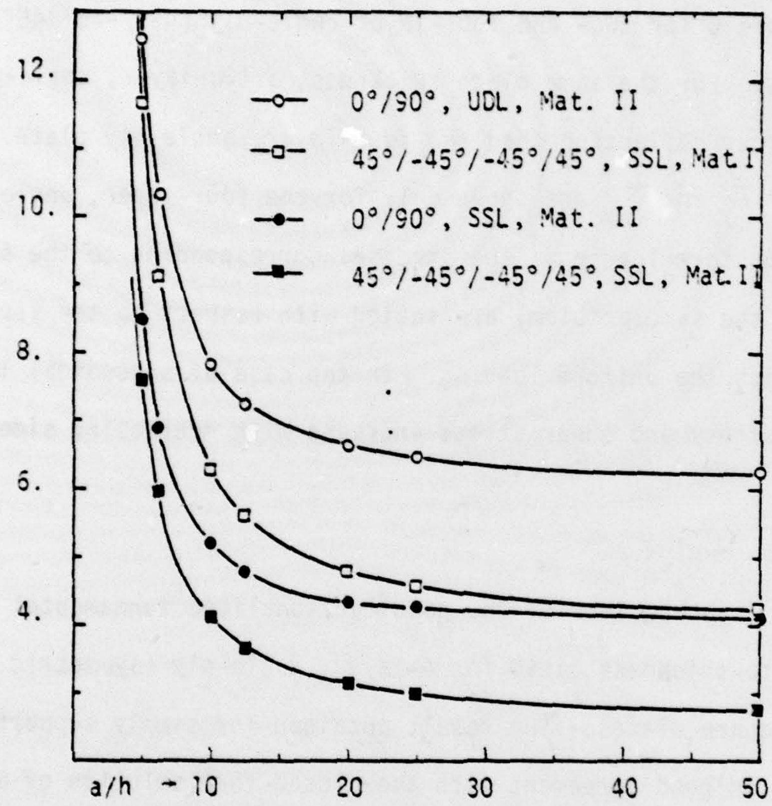


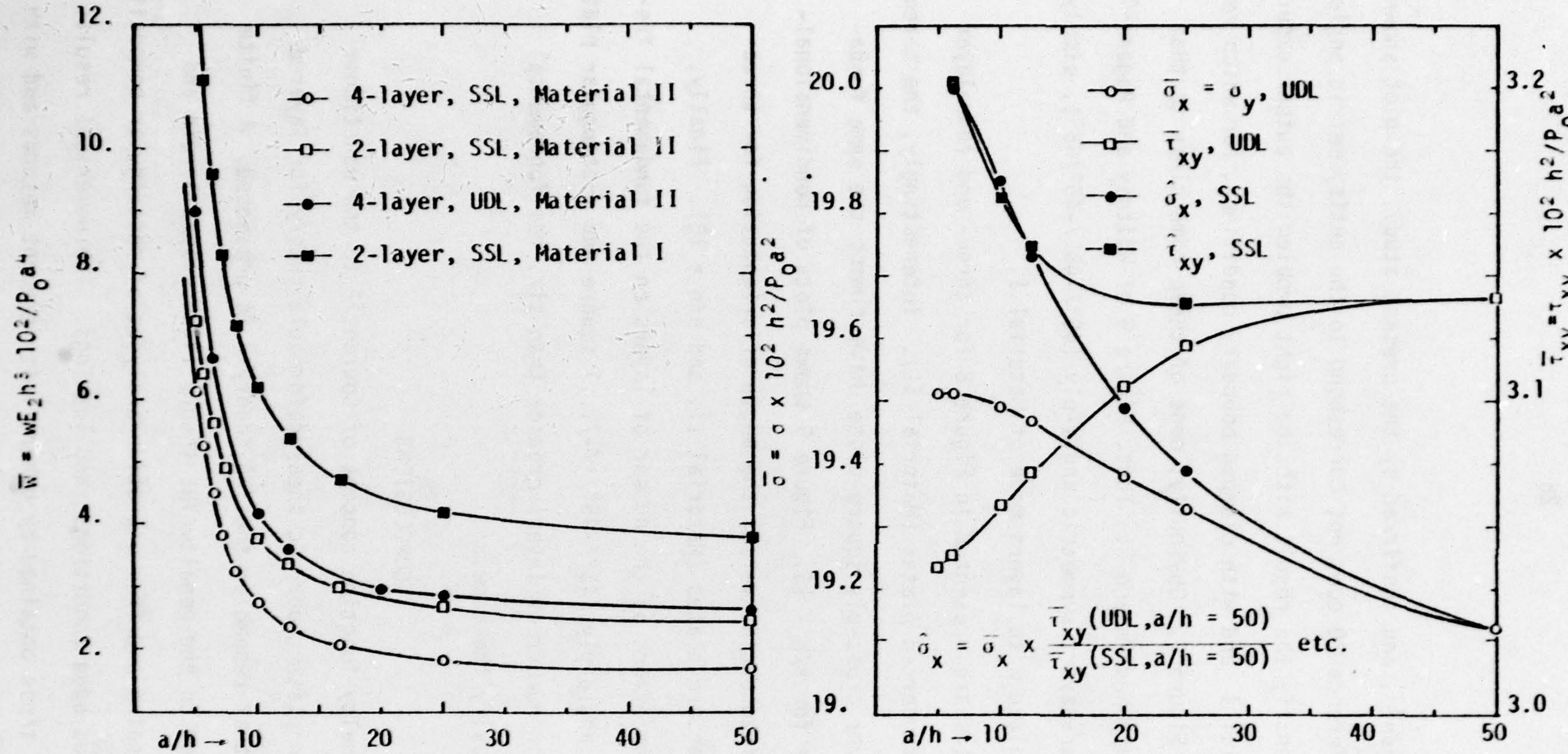
Figure 5 Bending deflections vs. side-to-thickness ratio for two-layer cross-ply (0°/90°) and four-layer, angle-ply, simply supported square plate

(45°/-45°/-45°/45°) square plate (Material I and II) under sinusoidal loading. It is clear that the effect of shear deformation is quite significant in (cross-ply, as well as angle-ply) composites with side-to-thickness ratio, $a/h < 20$.

Bending deflections and stresses versus side-to-thickness ratio are shown in Figure 6 for two- and four-layer angle-ply (45°/-45°/45°//45°) square plates. For the same plate thickness, a two-layer, angle-ply plate undergoes larger deflection than the four-layer, angle-ply plate. The stresses $\bar{\sigma}_x = \bar{\sigma}_y$ and $\bar{\tau}_{xy}$ are shown only for the four-layer, angle-ply plate under uniform loading. The stresses corresponding to the sinusoidal loading (for the same problem) are scaled with respect to the stresses associated with the uniform loading. In the case of sinusoidal loading both normal stress and shear stress increase with decreasing side to thickness ratio.

Free vibration analysis

Figure 7 shows plots of the nondimensionalized fundamental frequencies versus side-to-thickness ratio for 4-layer, angle-ply (symmetric and antisymmetric) square plates. The result obtained for simply supported (Material II) plate is in good agreement with the closed-form solution of Bert and Chen[23]. The present study predicts higher frequencies, with the deviation increasing with a/h. Figure 7 also shows the plot of fundamental frequencies for the symmetric angle-ply (45°/-45°/-45°/45°, Material I). Incidentally, this plot is in excellent agreement with that in Figure 5 of Whitney and Pagano[20]. However, the figure caption there (i.e. in reference 20) says that the result was obtained for four-layer, antisymmetric angle-ply (45°/-45°/45°/-45°), simply-supported square plate (Material II). As pointed



(a) Deflections

(b) Stresses (4-layer, Material II)

Figure 6 Bending deflections and stresses vs. side-to-thickness ratio for angle-ply, simply supported square plates

out by Bert and Chen[23], and confirmed by the present study, the plot shown in Figure 5 of Reference 20 does not correspond to the antisymmetric angle-ply plate. To identify the result with the right problem the author experimented with Material I and with clamped boundary conditions, for which results are also shown in Figure 7. Obviously, none of these come close to that presented by Whitney and Pagano[20]. Thus, Figure 5 of Whitney and Pagano[20] corresponds to four-layer, symmetric angle-ply (45°/-45°/-45°/45°), simply supported square plate with layers made of Material I.

Similar results are presented in Figure 8 for three- and four-layer cross-ply simply supported plates (Material II). Interestingly, the three-layer and four-layer cross-ply square plate have almost the same fundamental frequencies for $a/h < 15$. Figure 9 shows plots of nondimensionalized fundamental frequencies versus the angle of orientation for three- and four-layered square plates (Material II, and $a/h = 10$). Finally, Figure 10 shows the effect of the number of layers on the fundamental frequency of layered angle-ply (45°/-45°/+/-/...) square and rectangular plates (Material II). For number of layers greater than six, the fundamental frequency is virtually the same.

CONCLUSIONS

Using the penalty function concept of Courant[35] to the equations governing the thin-plate theory, a shear deformable theory for layered composite plates that resembles the YNS theory[12] is presented. A finite element model based on the penalty/YNS theory is developed herein and applied to the bending and free vibration analyses of rectangular composite plates with various edge conditions and loadings. The numerical results are compared with those obtained by other finite-element methods and with

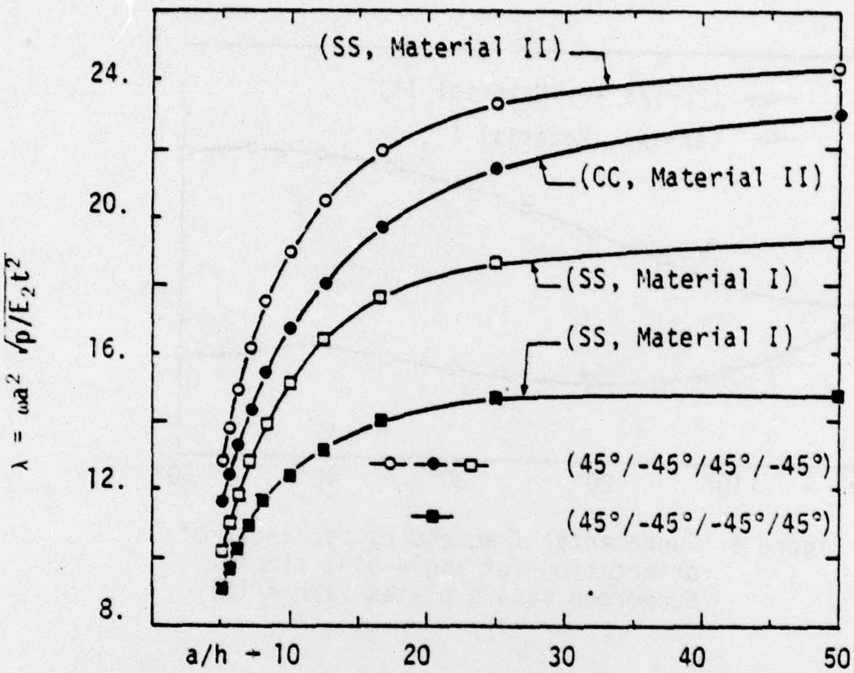


Figure 7 Fundamental frequencies vs. side-to-thickness ratio for 4-layer, angle-ply square plates

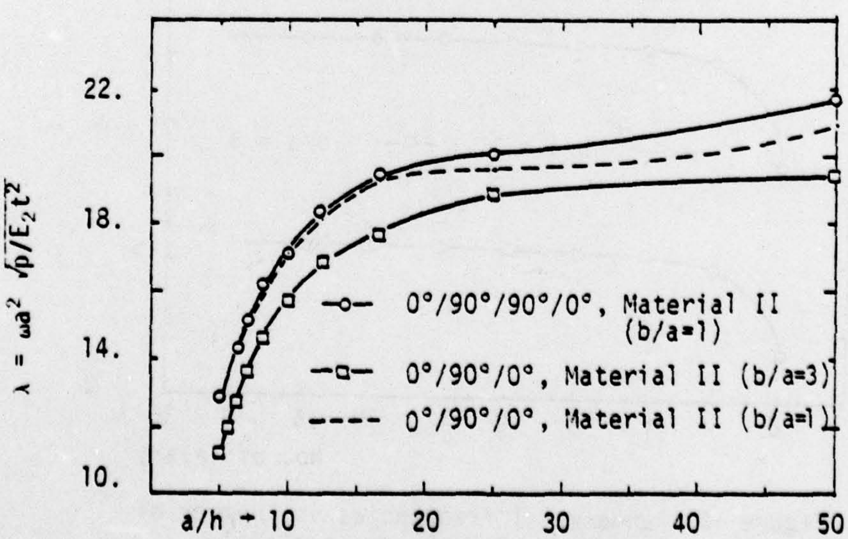


Figure 8 Fundamental frequencies vs. side-to-thickness ratio for cross-ply plates

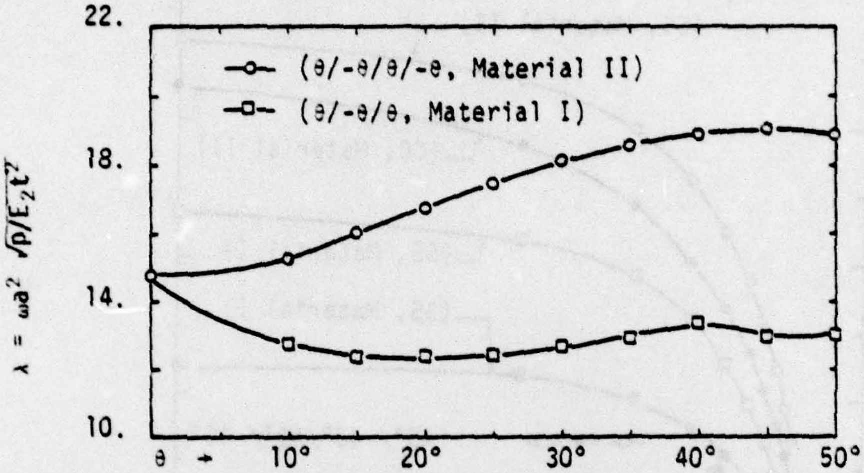


Figure 9 Fundamental frequencies vs. angle of orientation for angle-ply, simply supported square plates (a/h = 10)

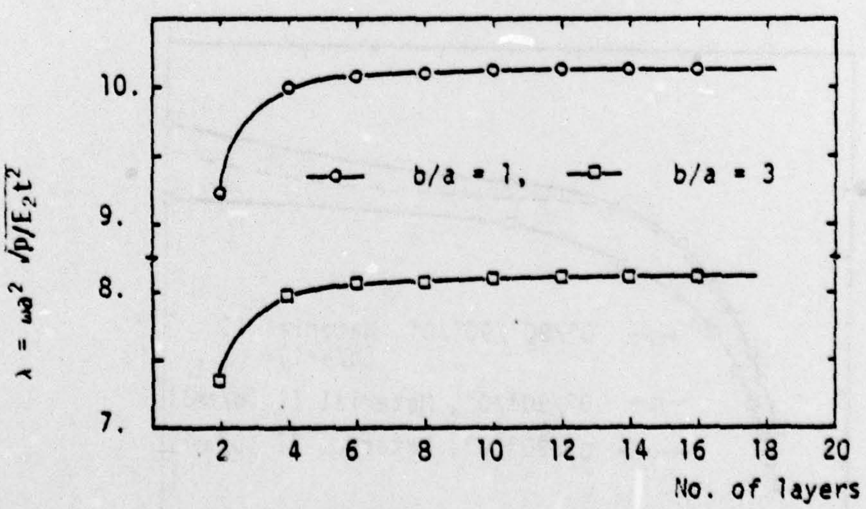


Figure 10 Fundamental frequencies vs. number of layers for four-layer, angle-ply (45°/-45°/45°/-45°), simply supported plate (a/h = 10, Material II)

exact solutions. The present element, despite its simplicity in formulation and programming, gives the most accurate results.

Application of the element to nonlinear (in von Karman sense) and bimodulus (i.e. different elastic properties in tension and compression) plate problems was investigated recently by the author[44,45]. However, its application to a nonlinear, shear deformable theory of composite plates is still awaiting.

Acknowledgements

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APPENDIX

A_{ij}, B_{ij}, D_{ij} = extensional, flexural-extensional, and flexural stiffnesses ($i,j=1,2,6$)

a,b = plate planform dimensions in x, y directions

E_1, E_2 = layer elastic moduli in directions along fibers and normal to them, respectively

F_i = force components in the finite element formulation ($i=1,2,\ldots,5$)

G_{12}, G_{13}, G_{23} = layer in-plane and thickness shear moduli

h = total thickness of plate

I = rotary inertia coefficient per unit midplane area of lamina

k_i = shear correction coefficients associated with the yz and xz planes, respectively ($i=1,2$)

$K_{ij}^{\alpha\beta}$ = element stiffness coefficients ($i,j=1,2,\ldots,80$; $\alpha,\beta=1,2,\ldots,5$)

L = total number of layers in the plate

M_i, N_i = stress couple, and stress resultant, respectively ($i=1,2,6$)

M_{ij}^e = element mass coefficients ($i,j=1,2,\ldots,8$)

N_i^e = element shape functions ($i=1,2,\ldots,8$)

n = nodes per element

ρ = laminate normal inertia coefficient per unit midplane area

P = transversely distributed load

P_o = intensity of transversely distributed load

Q_x, Q_y = shear stress resultants

Q_{ij} = plane stress reduced stiffness coefficients $(i,j=1,2,6)$

R = laminate rotary-normal coupling inertia coefficient per unit midplane area

$S_{ij}^{\xi\eta}$ = element matrices in FEM formulation $(i,j=1,2,\ldots,8;\ \xi,\eta=0,x,y)$

t = time

u,v,w = displacement components in x, y, z directions, respectively

u_o, v_o = in-plane displacements in x, y directions

u_i, v_i = nodal values of displacements u, v $(i=1,2,\ldots,8)$

U, U_1, U_2, U_L = strain energies

V = potential energy

x,y,z = position coordinates in cartesian system

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ = shear strains

$\{\Delta\}$ = column of vector of generalized nodal displacements

ε_i = penalty parameters $(i=1,2)$

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ = normal strains

θ_m = orientation of m-th laminate $(m=1,2,\ldots,L)$

λ_x, λ_y = Lagrange multipliers

π, π_p = total potential energy functionals

$\sigma_x, \sigma_y, \sigma_z$ = normal stresses

$\tau_{xy}, \tau_{xz}, \tau_{yz}$ = shear stresses

ψ_x, ψ_y = slope functions

ω = fundamental frequency of free vibration

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